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Symplectic and Nonsymplectic Realizations of Groups for Singular Systems. An Example

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The actions of groups for singular systems are studied in the framework of the theory of canonical transformations for presymplectic systems. Symplectic realizations as well as nonsymplectic ones arise in a natural way. As a typical example we construct the Poincaré realizations for the relativistic free massive particle.

1. INTRODUCTION

The study of constrained dynamical systems is a subject of interest because all the theories exhibiting gauge invariance are constrained. In a previous work (Cariñena *et al.*, 1985) the theory of canonical transformations for these systems was developed. This theory introduces the concept of an equivalence class of canonical transformations and suggests studying the actions of Lie groups on these systems in this framework.

In this paper we explain this method briefly and we apply it to construct the Poincaré realizations of a free particle.

2. GENERAL REMARKS

Let us start by recalling the theory of actions of a Lie group on a symplectic manifold (Abraham and Marsden, 1978; Giachetti, 1981; Cariñena and Ibort, 1985). Let (P, Ω) be a symplectic manifold, G a Lie group, and $\mathcal{G}$ its associated Lie algebra. We call an *action* of the group G on the manifold P a homomorphism $\Phi: G \to \mathrm{Diff}(P)$. The action is said to be *symplectic* iff it preserves the symplectic structure Ω; that is, $\Phi_g^*\Omega = \Omega$, $\forall g \in G$. Then we say that G acts symplectically on P. Every action induces

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a Lie algebra homomorphism $X:\mathscr{G}\to\mathscr{X}(P)$, by defining

$$X_g(f)(x):=\frac{d}{dt}f(\exp\{-tg\}x)\big|_{t=0},\qquad \forall x\in P,\quad \forall f\in C^\infty(P)$$

Furthermore, if the action is symplectic, then $X:\mathscr{G}\to\mathscr{X}_{\rm lH}(P)$, where $\mathscr{X}_{\rm lH}(P)$ denotes the set of locally Hamiltonian vector fields (l.H.v.f.) in P, and conversely.

In recent papers (Gomis *et al.*, 1984; Cariñena *et al.*, 1985) the theory of canonical transformations (c.t.) for presymplectic systems has been developed. Thus, let (C,ω) be the final constraint submanifold (f.c.s.) of a constrained dynamical system, and (P,Ω) an ambient (symplectic) manifold where the f.c.s. is embedded (which could be the "initial phase space" of the system or not).[2] Then a *canonical transformation* for such a system is defined as a pair of diffeomorphisms (ϕ,φ), where φ is identified with a *presymplectomorphism* on the f.c.s. of the dynamical system,[3] and ϕ is any extension of φ to P. Now, an equivalence relation is established in the set of these c.t.: two c.t. are related iff they have the same restriction φ on the f.c.s. Then it is proved that there exists a representant of each equivalence class of c.t. such that ϕ is a symplectomorphism in P. In addition, the set $\operatorname{Ker}\omega:=\{Z\in\mathscr{X}(C)\,|\,i(Z)\omega=0\}$ is closed under the Lie bracket operation and then generates a foliation F_ω of C. The quotient space $\tilde C:=C/F_\omega$ inherits a symplectic structure $\tilde\Omega\in Z^2(C)$ from (C,ω) [we assume (C,ω) is a sufficiently nice manifold to assure that C has, in turn, the structure of a differential manifold (Lichnerowicz, 1975)]. Therefore, it can be proved also that all the c.t. belonging to the same equivalence class $[\varphi]$ reduce to a unique symplectomorphism $\tilde\varphi$ in the symplectic manifold $(\tilde C,\tilde\Omega)$.

Next we can study the action of Lie groups for presymplectic systems, using the previous ideas. If (C,ω) is a presymplectic manifold, we say $\varphi: G\to \operatorname{Diff}(C)$ is a *presymplectic action* iff ω is φ-invariant; that is, $\varphi_g^*\omega=\omega$, with $\varphi_g:=\varphi(g)$, $\forall g\in G$. Since this is equivalent to $L(X_g)\omega=0$, $\forall g\in G$, we have that every presymplectic action induces a Lie algebra homomorphism $X:\mathscr{G}\to\mathscr{X}_{\rm lH}(C)$, where

$$\mathscr{X}_{\rm lH}(C):=\{Y\in\mathscr{X}(C)\,|\,i(Y)\omega\in Z^1(C)\ \text{(closed one-forms)}\}$$

is the set of presymplectic l.H.v.f. in (C,ω) (Gomis *et al.*, 1984). The connection with the theory of c.t. is carried out using the fact that a vector field in C is a presymplectic l.H.v.f. if and only if its flux generates an uniparametric local group of diffeomorphisms which are presymplectomorphisms of (C,ω) (Cariñena *et al.*, 1985). In addition, it can be proved

[2]That is, (P,C,Ω) is a *regular canonical system* in the terminology of Sniatycki (1974).
[3]That is, a transformation leaving invariant the presymplectic structure of this submanifold.

that every presymplectic action of G in C reduces to a symplectic action of G in the symplectic manifold $(\tilde{C}, \tilde{\Omega})$.

On the other hand, if (P, C, Ω) is a regular canonical system, the immediate application of the theory of c.t. for constrained systems leads one to define the action of the Lie group G on the system (P, C, Ω) as a pair of diffeomorphisms (ϕ, φ), where ϕ and φ are actions of G in P and C, respectively, and such that

$$\phi_g \circ j_C = j_C \circ \varphi_g, \quad \forall g \in G$$

where $j_C : C \hookrightarrow P$ is the embedding of C in P. In an analogous way as for the set of c.t. of the system (P, C, Ω), an equivalence relation is established in the set of actions of G in (P, C, Ω). Moreover, if φ is a presymplectic action in C, then, among all the elements (ϕ_i, φ) pertaining to the equivalence class $[\varphi]$, a representative (Φ, φ) can be chosen such that Φ is a symplectic action in P. In any case, all the actions belonging to the same equivalence class $[\varphi]$ reduce to the same (unique) symplectic action $\tilde{\varphi}$ in the reduced manifold $(\tilde{C}, \tilde{\Omega})$.

3. AN EXAMPLE: THE RELATIVISTIC FREE PARTICLE

Now we apply these results in order to construct the Poincaré realizations of a free particle (with mass m). First we describe the geometrical characteristics of the system [a more detailed discussion can be found in Cariñena *et al.* (1987).

The configuration space Q of this system is the Minkowski space. Then the momentum phase space T^*Q is an eight-dimensional manifold with local coordinates (q^μ, p_μ), $\mu = 0, 1, 2, 3$, which is endowed with the natural symplectic form $\Omega = dq^\mu \wedge dp_\mu$. The f.c.s. is the submanifold $j_S : S \hookrightarrow T^*Q$ locally defined by the constraint $\zeta = p_\mu p^\mu - m^2$, which can be written in the form $\zeta = p_0 \mp (p_i p^i + m^2)^{1/2}$ $(i = 1, 2, 3)$ and it makes clear that S is the union of two disconnected components, $S = S^- \cup S^+$. Each one inherits from (T^*Q, Ω) a presymplectic structure, locally given by

$$\omega^\mp = \left(\frac{p_i}{\mp (p_k p^k + m^2)^{1/2}} \, dq^0 + dq^i \right) \wedge dp_i$$

The generalized Darboux theorem (Abraham and Marsden, 1978) assures that in S there exist sets of "presymplectic coordinates" $(\hat{q}^\mu, \hat{p}_i)$ such that

$$\omega^\mp = d\hat{q}^\mu \wedge d\hat{p}_i$$

and it is also possible to complete these sets in such a manner that $(\hat{q}^\mu, \hat{p}_\mu)$ are local sets of canonical coordinates in (T^*Q, Ω). A transformation

ψ: $(q^\mu, p_\mu) \to (\hat{q}^\mu, \hat{p}_\mu)$ giving this change explicitly is (Dominici *et al.*, 1981)

$$
\begin{aligned}
&(\psi^*\hat{q}^0)^\mp = q^0, \qquad (\psi^*\hat{p}_0)^\mp = p_0 \mp (p_i p^i + m^2)^{1/2} \\
&(\psi^*\hat{q}^i)^\mp = q^i \pm \frac{q^0 p^i}{(p_k p^k + m^2)^{1/2}}, \qquad (\psi^*\hat{q}^0)^\mp = p_i
\end{aligned}
\tag{3.1}
$$

Besides (T^*Q, Ω), another ambient symplectic manifold (P, Ω_P) containing (S, ω) can be constructed by applying the coisotropic embedding theorem (Gotay, 1982; Marle, 1983). Since S is the union of the two disconnected components S^- and S^+, we have that this ambient manifold is made also of two symplectic components denoted (P^-, Ω_P^-) and (P^+, Ω_P^+). As local sets of symplectic coordinates for $(P, \Omega_P)^\mp$ we take $(\hat{q}^\mu, \hat{p}_\mu)^\mp$, and then the relations (3.1) are also explicit expressions of the local symplectomorphism ψ: $(T^*Q, \Omega) \to (P, \Omega_P)^\mp$ connecting both canonical systems.

4. REALIZATIONS OF THE POINCARÉ GROUP FOR THE FREE PARTICLE

In order to give a realization of the Poincaré group for this dynamical system, we proceed in two steps: first we give a realization in the f.c.s. (in particular, on each component) and afterward we extend it to the ambient symplectic manifold.

Let us start by studying the realizations in the component (S^-, ω^-). The infinitesimal generators of the Poincaré group have to be represented by a presymplectic l.H.v.f. and this implies that the functions M^μ_ν, $P_\mu \in C^\infty(S^-)$ have to be presymplectic locally Hamiltonian functions for those fields, that is,

$$
\begin{aligned}
i(X^\mu_\nu)\omega^- &= dM^\mu_\nu \\
i(Y_\mu)\omega^- &= dP_\mu
\end{aligned}
\tag{4.1}
$$

It is proved (Gomis *et al.*, 1984; Cariñena *et al.*, 1985) that the necessary and sufficient condition for a system like (4.1) to have solution is that the presymplectic locally Hamiltonian functions M^μ_ν, P_μ must be invariant under the action of Ker ω^-. This condition is locally equivalent to demanding that the presymplectic locally Hamiltonian functions cannot depend on the "gauge coordinates," in this case $\{\hat{q}^0\}$.

The form of the presymplectic coordinates, as well as physical considerations, suggest we take the following natural realization in (S^-, ω^-):

$$
\begin{aligned}
M^i_j &= \hat{q}^i \hat{p}_j - \hat{q}_j \hat{p}^i, \qquad P_i = \hat{p}_i \\
M^i_0 &= \hat{q}^i (\hat{p}_k \hat{p}^k + m^2)^{1/2}, \qquad P_0 = (\hat{p}_k \hat{p}^k + m^2)^{1/2}
\end{aligned}
$$

We can observe that in the reduced manifold $(\tilde{S}^-, \tilde{\Omega}^-)$ and referred to the local chart of symplectic coordinates $(\tilde{q}^i, \tilde{p}_i)$, such that $\tilde{q}^i = \hat{q}^i \circ \rho$, $\tilde{p}_i = \hat{p}_i \circ \rho$, (where $\rho: S^- \to \tilde{S}^-$ is the projection), this realization reduces to

$$\tilde{M}^i_j = \tilde{q}^i\tilde{p}_j - \tilde{q}_j\tilde{p}^i, \qquad \tilde{P}_i = \tilde{p}_i$$

$$\tilde{M}^i_0 = \tilde{q}^i(\tilde{p}_k\tilde{p}^k + m^2)^{1/2}, \qquad \tilde{P}_0 = (\tilde{p}_k\tilde{p}^k + m^2)^{1/2}$$

which, as can be observed, is closed under the Poisson bracket $\{,\}_{\tilde{\Omega}^-}$ and hence it is a symplectic realization of the Poincaré group in $(\tilde{S}^-, \tilde{\Omega}^-)$. Now we want to extend the realization to the ambient symplectic component (P^-, Ω^-_P). It is performed by taking the *extended generating functions* (Cariñena *et al.*, 1985)

$$\mathcal{M}^\mu_\nu := \mathbf{M}^\mu_\nu + \hat{\lambda}^\mu_\nu \zeta$$

$$\mathcal{P}_\mu := \mathbf{P}_\mu + \hat{\eta}_\mu \zeta$$

where $\hat{\lambda}^\mu_\nu$, $\hat{\eta}_\mu \in C^\infty(P^-)$ are arbitrary functions and $\mathbf{M}^\mu_\nu$, $\mathbf{P}_\mu \in C^\infty(P^-)$ are extensions of M^μ_ν, P_μ to P^-. Therefore, in (P^-, Ω^-_P) we obtain the following family of equivalent representations:

$$\mathcal{M}^i_j = \hat{q}^i\hat{p}_j - \hat{q}_j\hat{p}^i + \hat{\lambda}^i_j(\hat{q}^\mu, \hat{p}_\nu)\hat{p}_0$$

$$\mathcal{M}^i_0 = \hat{q}^i(\hat{p}_k\hat{p}^k + m^2)^{1/2} + \hat{\lambda}^i_0(\hat{q}^\mu, \hat{p}_\nu)\hat{p}_0$$

$$\mathcal{P}_i = \hat{p}_i + \hat{\eta}_i(\hat{q}^\mu, \hat{p}_\nu)\hat{p}_0$$

$$\mathcal{P}_0 = (\hat{p}_k\hat{p}^k + m^2)^{1/2} + \hat{\eta}_0(\hat{q}^\mu, \hat{p}_\nu)\hat{p}_0$$

whereas in the original ambient manifold (T^*Q, Ω) and referred to the chart (q^μ, p_μ), we have, taking into account (3.1),

$$\begin{aligned}
\mathcal{M}^i_j &= \left(q^i + \frac{q^0p^i}{(p_kp^k + m^2)^{1/2}}\right)p_j - \left(q_j + \frac{q^0p_j}{(p_kp^k + m^2)^{1/2}}\right)p^i \\
&\quad + \lambda^i_j(q^\mu, p_\nu)[p_0 - (p_ip^i + m^2)^{1/2}] \\
&= q^ip_j - q_jp^i + \lambda^i_j(q^\mu, p_\nu)[p_0 - (p_ip^i + m^2)^{1/2}] \\
\mathcal{M}^i_0 &= \left(q^i + \frac{q^0p^i}{(p_kp^k + m^2)^{1/2}}\right)(p_kp^k + m^2)^{1/2} \\
&\quad + \lambda^i_0(q^\mu, p_\nu)[p_0 - (p_ip^i + m^2)^{1/2}] \\
\mathcal{P}_i &= p_i + \eta_i(q^\mu, p_\nu)[p_0 - (p_ip^i + m^2)^{1/2}] \\
\mathcal{P}_0 &= (p_kp^k + m^2)^{1/2} + \eta_0(q^\mu, p_\nu)[p_0 - (p_ip^i + m^2)^{1/2}]
\end{aligned}$$

The functions $\mathcal{M}^\mu_\nu$, $\mathcal{P}_\mu$ are not a closed set under the corresponding Poisson bracket, because of the arbitrariness of the functions λ^μ_ν, η_μ. This means that the elements of this class of equivalent representations of the Poincaré group are not, in general, symplectic actions in (P^-, Ω_P^-) or in (T^*Q, Ω). Nevertheless, we can always choose the arbitrary functions in such a manner that the corresponding representation is a symplectic action. In fact, if we take

$$\hat{\lambda}^i_j = 0, \qquad \hat{\lambda}^i_0 = \hat{q}^i - \frac{\hat{q}^0 \hat{p}^i}{(\hat{p}_k \hat{p}^k + m^2)^{1/2}}$$

$$\hat{\eta}_i = 0, \qquad \hat{\eta}_0 = 1$$

or, equivalently,

$$\lambda^i_j = 0, \qquad \lambda^i_0 = q^i$$

$$\eta_i = 0, \qquad \eta_0 = 1$$

the symplectic realization (in the original coordinates) is

$$\mathcal{M}^i_j = q^i p_j - q_j p^i, \qquad \mathcal{P}_i = p_i$$

$$\mathcal{M}^i_0 = q^i p_0 - q_0 p^i, \qquad \mathcal{P}_0 = p_0$$

which is the standard kinematic realization of the Poincaré group for the free particle (Dirac, 1949; Sudarshan and Mukunda, 1974).

In an analogous way we construct the realizations starting from (S^+, ω^+).

5. CONCLUSIONS

Summing up, starting from a natural representation which is a presymplectic action of a group in the f.c.s. of a constrained dynamical system, we can extend it to any ambient symplectic manifold of this submanifold. This extension is not unique and we actually obtain a large family of equivalent representations which are not necessarily symplectic actions, although in every equivalence class the existence of a unique representative which is a symplectic action is assured.

It is evident that in the set of equivalent representations the more interesting one is just this symplectic action (for instance, in order to perform the geometric quantization of the system). In relation to this topic, a very extensive analysis of the free particle can be found, for instance, in Souriau (1970) and Simms and Woodhouse (1976).

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